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Identification Problems in Linear Elasticity

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Let Ω be a bounded domain in R^n with C^2-boundary and let D be a Lipschitz domain with $\overline{D} \subset \Omega$. We consider the inverse problem (determining D) to the system of linear elasticity

$$D_i\Big(\big(\mu_D(\delta_{ij}\delta_{rs} + \delta_{ir}\delta_{js}) + \lambda_D\delta_{ir}\delta_{js}\big)D_j u^s\Big) = 0 \qquad \text{in } \Omega,$$

where $\mu_D = \tilde{\mu}\chi_D + \mu\chi_{R^n\setminus D}$ and $\lambda_D = \tilde{\lambda}\chi_D + \lambda\chi_{R^n\setminus D}$. Under the condition on the Lamè constants $(\lambda - \tilde{\lambda})(\mu - \tilde{\mu}) \geq 0$, we show that D is uniquely determined by the complete knowledge of the Dirichlet-to-Neumann map. We also obtain an uniqueness result for the monotone case from one boundary measurement. © 1997 Academic Press

0. INTRODUCTION

Let Ω be a bounded domain in R^n, $n \geq 3$, with C^2-boundary and let D be a Lipschitz domain with $\overline{D} \subset \Omega$.

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We consider the Dirichlet problem of the system of linear elasticity,

$$\begin{cases} (L_D\mathbf{u})_r = \sum_{i,j,s=1}^{n} D_i\left(\left(a_{ij}^{rs}\chi_{R^n\setminus D} + b_{ij}^{rs}\chi_D\right)D_j u^s\right) = 0 \qquad \text{in } \Omega \\ \text{for each } r = 1,\dots,n \qquad \text{and} \qquad \mathbf{u}|_{\partial\Omega} = \boldsymbol{\varphi} \in H^{1/2}(\partial\Omega), \end{cases} \tag{0.1}$$

where

$$a_{ij}^{rs} = \mu(\delta_{ij}\delta_{rs} + \delta_{is}\delta_{jr}) + \lambda\delta_{ir}\delta_{js} \qquad \text{and}$$

$$b_{ij}^{rs} = \tilde{\mu}(\delta_{ij}\delta_{rs} + \delta_{is}\delta_{jr}) + \tilde{\lambda}\delta_{ir}\delta_{js}$$

for some constants μ, λ, $\tilde{\mu}$, and $\tilde{\lambda}$. We assume that the above constants satisfy the conditions

$$\mu > 0, \qquad \tilde{\mu} > 0, \qquad 2\mu + n\lambda > 0, \qquad 2\tilde{\mu} + n\tilde{\lambda} > 0 \tag{0.2}$$

and

$$|\mu - \tilde{\mu}| + |\lambda - \tilde{\lambda}| > 0. \tag{0.3}$$

We may consider Ω as an isotropic inhomogeneous linear elastic body with Lamé constants λ_D and μ_D, where $\mu_D = \tilde{\mu}\chi_D + \mu\chi_{R^n\setminus D}$ and $\lambda_D = \tilde{\lambda}\chi_D + \lambda\chi_{R^n\setminus D}$.

Note that due to the condition (0.2), the Lamé constants satisfy

$$0 < \delta < \mu_D, \qquad 2\mu_D + n\lambda_D < M \qquad \text{in } \Omega \tag{0.4}$$

for some constants δ and M. Hence it follows from the standard theory of linear elasticity that for every $\boldsymbol{\varphi} \in H^{1/2}(\partial\Omega)$, there exists a unique solution to the problem (0.1) (see [9]). The Dirichlet-to-Neumann map Λ_D corresponding to D is defined by

$$\Lambda_D\boldsymbol{\varphi} = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^t)N + \lambda(\operatorname{div}\mathbf{u})N \qquad \text{on } \partial\Omega,$$

where $\mathbf{u}$ is the solution to (0.1) and N is the unit outer normal to $\partial\Omega$ (see [9]).

In this paper we shall study the identification problem of D from the knowledge of Λ_D: that is, the problem to be considered here is whether we can determine the unknown subdomain D of Ω from the knowledge of a part of the Dirichlet-to-Neumann map Λ_D. This problem is an analogue of the inverse conductivity problem studied by several authors (for example, see [1, 7, 8, 10]) and especially by V. Isakov [6]. We shall show by extending the idea of V. Isakov in [6] that the whole knowledge of Λ_D determines D

uniquely under some condition on the Lamè constants (see Theorem 0.1). Also we show that in the monotone case there exists $\boldsymbol{\varphi} \in H^{1/2}(\partial\Omega)$ such that $\Lambda_D \boldsymbol{\varphi}$ suffices to determine D (see Theorem 0.2).

THEOREM 0.1. *Assume that the Lamè constants satisfy the condition*

$$(\lambda - \tilde{\lambda})(\mu - \tilde{\mu}) \geq 0. \tag{0.5}$$

Let D_1 and D_2 be Lipschitz domains such that for each i, $\overline{D}_i \subset \Omega$ and $\Omega \setminus \overline{D}_i$ is connected. Then

$$\Lambda_{D_1} = \Lambda_{D_2} \qquad \textit{implies} \qquad D_1 = D_2.$$

Let Ψ denote the space of all vector valued functions $\boldsymbol{\psi}$ on R^n satisfying the system of linear equations $D_i \psi_j + D_j \psi_i = 0$ for $1 \leq i, j \leq n$, and we define

$$\Psi_0 = \{\boldsymbol{\psi}|_{\partial\Omega} : \boldsymbol{\psi} \in \Psi\}$$

and

$$\Phi = \left\{\mathbf{u}|_{\partial\Omega} : \mathbf{u} \in H^1(\Omega), \Delta\mathbf{u} = 0, \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\right\}.$$

We remark that for $n = 3$, any $\boldsymbol{\psi} \in \Psi$ can be written as $\boldsymbol{\psi}(X) = a + b \times X$ for some $a, b \in R^3$ (see [11]).

Then for the one measurement problem we obtain the following uniqueness result:

THEOREM 0.2. *Assume that the Lamé constants satisfy the condition* (0.5). *Let D_1 and D_2 be Lipschitz domains such that for each i, $\overline{D}_i \subset \Omega$ and $\Omega \setminus \overline{D}_i$ is connected. Assume further that $D_1 \subset D_2$ or $D_2 \subset D_1$. Then*

$$\Lambda_{D_1}\boldsymbol{\varphi} = \Lambda_{D_2}\boldsymbol{\varphi} \qquad \textit{for some } \varphi \in H^{1/2}(\partial\Omega) \setminus \Phi \qquad \textit{implies} \qquad D_1 = D_2.$$

When $\mu \neq \tilde{\mu}$, the above statement holds for $\boldsymbol{\varphi} \in H^{1/2}(\partial\Omega) \setminus \Psi_0$.

Note that if $\mathbf{u} = (x_1, \ldots, x_n)$, then clearly $\mathbf{u}|_{\partial\Omega} \notin \Phi$. We shall prove Theorem 0.1 in Section 3 and Theorem 0.2 is proved in Section 4.

1. NOTATIONS AND DEFINITIONS

Throughout this paper, we denote by D a subdomain of Ω with Lipschitz boundary. If we set $c_D^{ijrs} = a_{ij}^{rs} \chi_{R^n \setminus D} + b_{ij}^{rs} \chi_D$, then by (0.4)

$$C^{-1}|\xi|^2|\eta|^2 \leq c_D^{ijrs} \xi_i \xi_j \eta^s \eta^r \leq C|\xi|^2|\eta|^2 \tag{1.1}$$

and

$$|\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 \le C\left(c_D^{ijrs} D_j u^s D_i u^r\right), \tag{1.2}$$

where $\nabla \mathbf{u}$ denotes the Jacobian matrix $(D_i u^j)$ of $\mathbf{u}$ and C is a positive constant depending only on δ and M. Henceforth C will denote general positive constants which may differ in each occurrence and we use the summation convention on repeated indices. Also we denote by $|E|$ the n-dimensional Lebesgue measure of a subset E of R^n. The following well-known Korn's inequality will be of use to us.

LEMMA 1.1. *If* $\mathbf{u} \in H_0^1(\Omega)$, *then*

$$\int_\Omega |\nabla \mathbf{u}|^2 \le \frac{1}{2} \int_\Omega |\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2.$$

In the course of our proof of Theorem 0.1 we need to construct a singular solution to $L_D \mathbf{u} = 0$ near $\overline{D}$. We accomplish this task by using the single layer potentials on ∂D corresponding to the system of linear elasticity with the Lamé constants μ, λ and $\tilde{\mu}$, $\tilde{\lambda}$, respectively. Define

$$\mathcal{S}_D \mathbf{f}(X) = \int_{\partial D} \Gamma(X - Q)\mathbf{f}\, d\sigma(Q) \qquad \left(\mathbf{f} \in L^2(\partial D)\right)$$

and

$$\tilde{\mathcal{S}}_D \mathbf{g}(X) = \int_{\partial D} \tilde{\Gamma}(X - Q)\mathbf{g}\, d\sigma(Q) \qquad \left(\mathbf{g} \in L^2(\partial D)\right),$$

where $\Gamma(X) = (\Gamma^{ij}(X))$ and $\tilde{\Gamma}(X) = (\tilde{\Gamma}^{ij}(X))$ are the fundamental solution matrices corresponding to the operators $\mu\Delta + (\lambda + \mu)\nabla\,\mathrm{div}$ and $\tilde{\mu}\Delta + (\tilde{\lambda} + \tilde{\mu})\nabla\,\mathrm{div}$, respectively.

For $n \ge 3$, the entries Γ^{ij} are given by

$$\Gamma^{ij}(X) = \frac{A}{(n-2)\,\omega_n}\delta_{ij}|X|^{2-n} + \frac{B}{\omega_n} X_i X_j |X|^{-n}$$

and

$$\tilde{\Gamma}^{ij}(X) = \frac{\tilde{A}}{(n-2)\,\omega_n}\delta_{ij}|X|^{2-n} + \frac{\tilde{B}}{\omega_n} X_i X_j |X|^{-n},$$

where

$$A = \frac{3\mu + \lambda}{2\mu(2\mu + \lambda)}, \qquad B = \frac{\mu + \lambda}{2\mu(2\mu + \lambda)}, \qquad \tilde{A} = \frac{3\tilde{\mu} + \tilde{\lambda}}{2\tilde{\mu}(2\tilde{\mu} + \tilde{\lambda})},$$

and $\tilde{B} = \dfrac{\tilde{\mu} + \tilde{\lambda}}{2\tilde{\mu}(2\tilde{\mu} + \tilde{\lambda})}$.

Then the following properties for $\mathcal{S}_D$ are well known,

$$\mu\Delta \mathcal{S}_D\mathbf{f} + (\lambda + \mu)\nabla(\operatorname{div} \mathcal{S}_D\mathbf{f}) = \mathbf{0} \qquad \text{in } R^n \setminus \partial D,$$
$$\mathcal{S}_D\mathbf{f} \text{ is continuous across } \partial D,$$

and

$$\|\nabla \mathcal{S}_D\mathbf{f}\|_{L^2(D)} + \|\nabla \mathcal{S}_D\mathbf{f}\|_{L^2(\Omega\setminus D)} \le C\|\mathbf{f}\|_{L^2(\partial D)}, \tag{1.3}$$

where C depends only on the Lipschitz character of D, λ, and μ. The conormal derivative $\partial/\partial\nu$ on ∂D associated to the system of elasticity with the Lamè constants μ and λ is given by

$$\frac{\partial \mathbf{u}}{\partial \nu} = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nu + \lambda(\operatorname{div}\mathbf{u})\nu \qquad \text{on } \partial D,$$

where ν is the unit outer normal to ∂D. Then from the results in [2]

$$\frac{\partial}{\partial\nu}\mathcal{S}_D\mathbf{f}^{\pm}(P) = \pm\frac{1}{2}\mathbf{f}(P) + \mathcal{K}^*\mathbf{f}(P) \qquad \text{a.e. } P \in \partial D,$$

where $\mathcal{K}^*$ is a bounded singular integral operator on $L^2(\partial D)$. Here the subscripts $+$ and $-$ indicate the nontangential limits taken inside D and outside $\overline{D}$, respectively (see [3, 2], for details). The same argument holds for the conormal derivative corresponding to the Lamé constants $\tilde{\mu}$, $\tilde{\lambda}$. Finally we state the following theorem in [4] which is essential to prove Theorem 0.1.

THEOREM 1.2. *Assume that the Lamé constants satisfy the condition* (0.5).

Then the mapping

$$(\mathbf{f}, \mathbf{g}) \mapsto \left(\mathcal{S}_D\mathbf{f} - \tilde{\mathcal{S}}_D\mathbf{g}, \frac{\partial}{\partial\nu}\mathcal{S}_D\mathbf{f}^- - \frac{\partial}{\partial\tilde{\nu}}\tilde{\mathcal{S}}_D\mathbf{g}^+ \right)$$

is an invertible operator of $L^2(\partial D) \times L^2(\partial D)$ *onto* $H^1(\partial D) \times L^2(\partial D)$.

2. PRELIMINARY LEMMAS

In this section we present some lemmas needed to prove Theorem 0.1. The proof of the following lemma is standard but for the sake of completeness we include its proof.

LEMMA 2.1. *Let* $\mathbf{u} \in H^1(\Omega)$ *be a solution to* $L_D \mathbf{u} = \mathbf{f} \in L^2(\Omega)$ *in* Ω. *Suppose that* $\{\mu_m\}$ *and* $\{\lambda_m\}$ *are sequences in* $L^\infty(\Omega)$ *such that*

(i) $|\{X \in \Omega : \mu_m(X) \neq \mu_D$ *or* $\lambda_m(X) \neq \lambda_D\}| \to 0$ *as* $m \to +\infty$ *and*

(ii) $0 < \delta_1 < \mu_m, 2\mu_m + n\lambda_m < M_1 < +\infty$ *in* Ω *for all* m.

Then if $\mathbf{u}_m \in H^1(\Omega)$ *is the solution to the Dirichlet problem*

$$\begin{cases} D_i\left(c_m^{ijrs} D_j u_m^s\right) = f^r & \text{in } \Omega \qquad \text{for all } r = 1, \ldots, n \\ \mathbf{u}_m = \mathbf{u} & \text{on } \partial\Omega, \end{cases}$$

where $c_m^{ijrs} = \mu_m(\delta_{ij}\delta_{rs} + \delta_{is}\delta_{jr}) + \lambda_m \delta_{ir}\delta_{js}$, *then*

$$\mathbf{u}_m \to \mathbf{u} \qquad \text{in } H^1(\Omega).$$

Proof. Taking $\mathbf{v}_m = \mathbf{u}_m - \mathbf{u} \in H_0^1(\Omega)$ as a test function, we obtain

$$\int_\Omega \mu_D(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{v}_m + \lambda_D(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{v}_m) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_m$$

and

$$\int_\Omega \mu_m\left(\nabla\mathbf{u}_m + \nabla\mathbf{u}_m^t\right)\nabla\mathbf{v}_m + \lambda_m(\operatorname{div}\mathbf{u}_m)(\operatorname{div}\mathbf{v}_m) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_m,$$

so that

$$\int_\Omega \mu_m\left(\nabla\mathbf{v}_m + \nabla\mathbf{v}_m^t\right)\nabla\mathbf{v}_m + \lambda_m(\operatorname{div}\mathbf{v}_m)^2$$
$$= \int_\Omega (\mu_D - \mu_m)(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{v}_m + (\lambda_D - \lambda_m)(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{v}_m).$$

By the estimate (1.2) together with hypothesis (ii), we can choose $\epsilon > 0$ so that

$$
\begin{aligned}
&\epsilon \int_\Omega |\nabla \mathbf{v}_m + \nabla \mathbf{v}_m^t|^2 \\
&\qquad \le \int_\Omega (\mu_D - \mu_m)(\nabla \mathbf{u} + \nabla \mathbf{u}^t)\nabla \mathbf{v}_m + (\lambda_D - \lambda_m)(\operatorname{div} \mathbf{u})(\operatorname{div} \mathbf{v}_m) \\
&\qquad \le \frac{2}{\epsilon} \int_\Omega (\mu_D - \mu_m)^2 |\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 + \frac{\epsilon}{2} \int_\Omega |\nabla \mathbf{v}_m|^2 \\
&\qquad\quad + \frac{2n}{\epsilon} \int_\Omega (\lambda_D - \lambda_m)^2 (\operatorname{div} \mathbf{u})^2 + \frac{\epsilon}{2n} \int_\Omega (\operatorname{div} \mathbf{v}_m)^2.
\end{aligned}
$$

Note that $(\operatorname{div} \mathbf{v}_m)^2 \le n|\nabla \mathbf{v}_m|^2$. Combining this estimate with Lemma 1.1, we obtain

$$
\begin{aligned}
&\epsilon \int_\Omega |\nabla \mathbf{v}_m|^2 \\
&\qquad \le \frac{2}{\epsilon} \int_\Omega (\mu_D - \mu_m)^2 |\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 + \frac{2n}{\epsilon} \int_\Omega (\lambda_D - \lambda_m)^2 (\operatorname{div} \mathbf{u})^2,
\end{aligned}
$$

which implies that $\nabla \mathbf{v}_m \to 0$. Now the conclusion follows immediately from the Poincarè inequality.

The next lemma says that solutions to $L_D \mathbf{u} = 0$ in Ω have a Runge approximation property. To prove this lemma, we follow the idea of R. Kohn and M. Vogelius in [8] closely.

LEMMA 2.2. *Let Ω' be a smooth domain with $\overline{D} \subset \Omega'$ and $\overline{\Omega}' \subset \Omega$, and let $\mathbf{u} \in H^1(\Omega')$ be a solution to $L_D \mathbf{u} = 0$ in Ω'. Then given any compact subset K of Ω' containing D, there exists a sequence $\{\mathbf{u}_m\}$ in $H^1(\Omega)$ such that* (i) $L_D \mathbf{u}_m = 0$ *and* (ii) $\mathbf{u}_m \to \mathbf{u}$ *in $H^1(K)$.*

Proof. We first show that if Ω'' is a smooth domain with $K \subset \Omega''$ and $\overline{\Omega''} \subset \Omega'$, there exists a sequence $\{\mathbf{u}_m\}$ in $H^1(\Omega)$ such that (i) $L_D \mathbf{u}_m = 0$ in Ω and (ii$'$) $\mathbf{u}_m \to \mathbf{u}$ in $L^2(\Omega'')$. To show this, it suffices to prove that $\mathbf{u}$ belongs to the closure of the subspace $\mathcal{K}$ of $L^2(\Omega'')$, where $\mathcal{K}$ denotes the space of functions in $L^2(\Omega'')$ which are restrictions to Ω'' of solutions $\mathbf{u} \in H^1(\Omega)$ to $L_D \mathbf{u} = 0$ in Ω. Hence by the Hahn–Banach theorem, we must show that if $\mathbf{v} \in L^2(\Omega'')$ satisfies $\int_{\Omega''} \mathbf{v} \cdot \mathbf{k}\, dx = 0$ for all $\mathbf{k} \in \mathcal{K}$, then $\int_{\Omega''} \mathbf{v} \cdot \mathbf{u}\, dx = 0$.

Suppose that $\mathbf{v} \in L^2(\Omega'')$ satisfies $\int_{\Omega''} \mathbf{v} \cdot \mathbf{k}\, dx = 0$ for all $\mathbf{k} \in \mathcal{K}$.

Define $\mathbf{V} \in L^2(\Omega)$ by

$$\mathbf{V} = \begin{cases} \mathbf{v} & \text{in } \Omega'' \\ 0 & \text{in } \Omega \setminus \Omega'' \end{cases}$$

and let $\mathbf{w} \in H^1(\Omega)$ be the solution to the Dirichlet problem

$$L_w \mathbf{w} = \mathbf{V} \text{ in } \Omega \qquad \text{and} \qquad \mathbf{w} = 0 \text{ on } \partial\Omega.$$

Then if $\mathbf{k} \in H^1(\Omega)$ is the solution to the Dirichlet problem

$$L_D \mathbf{k} = 0 \text{ in } \Omega \qquad \text{and} \qquad \mathbf{k} = \mu(\nabla\mathbf{w} + \nabla\mathbf{w}^t)N + \lambda(\operatorname{div}\mathbf{w})N \text{ on } \partial\Omega,$$

then

$$\begin{aligned}
&\int_{\partial\Omega} \left| \mu(\nabla\mathbf{w} + \nabla\mathbf{w}^t)N + \lambda(\operatorname{div}\mathbf{w})N \right|^2 d\sigma \\
&\quad = \int_{\partial\Omega} \left[\mu_D(\nabla\mathbf{w} + \nabla\mathbf{w}^t)N + \lambda_D(\operatorname{div}\mathbf{w})N \right] \cdot \mathbf{k}\, d\sigma \\
&\quad = \int_{\Omega} \mu_D(\nabla\mathbf{w} + \nabla\mathbf{w}^t)\nabla\mathbf{k} + \lambda_D(\operatorname{div}\mathbf{w})(\operatorname{div}\mathbf{k}) + \int_{\Omega} \mathbf{V}\cdot\mathbf{k} \\
&\quad = \int_{\Omega''} \mathbf{v}\cdot\mathbf{k} = 0
\end{aligned}$$

and therefore

$$\mu(\nabla\mathbf{w} + \nabla\mathbf{w}^t)N + \lambda(\operatorname{div}\mathbf{w})N = 0 \qquad \text{on } \partial\Omega.$$

Since

$$\mu\Delta\mathbf{w} + (\lambda + \mu)\nabla\operatorname{div}\mathbf{w} = 0 \qquad \text{in } \Omega\setminus\Omega'',$$

it follows from the unique continuation theorem that

$$\mathbf{w} = 0 \qquad \text{in } \Omega\setminus\Omega'',$$

and so we get

$$\begin{aligned}
\int_{\Omega''} \mathbf{v}\cdot\mathbf{u} &= \int_{\Omega'} \mathbf{V}\cdot\mathbf{u} \\
&= \int_{\partial\Omega'} \left[\mu_D(\nabla\mathbf{w} + \nabla\mathbf{w}^t)N + \lambda_D(\operatorname{div}\mathbf{w})N \right] \cdot \mathbf{u}\, d\sigma \\
&\quad - \int_{\Omega'} \mu_D(\nabla\mathbf{w} + \nabla\mathbf{w}^t)\nabla\mathbf{u} + \lambda_D(\operatorname{div}\mathbf{w})(\operatorname{div}\mathbf{u}) = 0.
\end{aligned}$$

This completes the proof that $\mathbf{u} \in \overline{K}$, the closure of K in $L^2(\Omega'')$.

Let $\{\mathbf{u}_m\}$ be a sequence in $H^1(\Omega)$ with the proper ties (i′) and (ii′). Then since $L_D(\mathbf{u}_m - \mathbf{u}) = 0$ in Ω', it follows from the Caccioppoli inequality (see [5, p. 76]) and Korn's inequality (Lemma 1.1) that

$$\int_K |\nabla \mathbf{u}_m - \nabla \mathbf{u}|^2 \leq C \int_{\Omega''} |\mathbf{u}_m - \mathbf{u}|^2.$$

Hence we have $\nabla \mathbf{u}_m \to \nabla \mathbf{u}$ in $L^2(K)$ and this completes the whole proof.

Remark. The unique continuation theorem for the elastic case seems to be not well known but its proof is identical to the Laplacian case and so omitted.

3. PROOF OF THEOREM 0.1

For simplicity, we will assume that $\Omega \setminus \overline{D_1 \cup D_2}$ is connected, leaving the details when $\Omega \setminus \overline{D_1 \cup D_2}$ is not connected for the readers (see Remark at the end of this section).

Assume that $D_1 \neq D_2$. Then we may assume that there exists an open ball $B_R = B_R(Q)$ of radius R, centered at $Q \in \partial D_1$ such that $\overline{B}_R \subset \Omega \setminus \overline{D}_2$. Set $D_3 = B_R \cup D_1$.

We first show the following claim:

Claim 1. If for each $j = 2, 3$, $\mathbf{u}_j$ is a solution to $L_{D_j}\mathbf{u}_j = 0$ in a vicinity of the set $\overline{D_1 \cup D_2}$, then

$$\int_{D_1} (\tilde{\mu} - \mu)\big(\nabla \mathbf{u}_3 + \nabla \mathbf{u}_3^t\big)\nabla \mathbf{u}_2 + (\tilde{\lambda} - \lambda)(\operatorname{div} \mathbf{u}_3)(\operatorname{div} \mathbf{u}_2)\, dx$$
$$= \int_{D_2} (\tilde{\mu} - \mu)\big(\nabla \mathbf{u}_3 + \nabla \mathbf{u}_3^t\big)\nabla \mathbf{u}_2 + (\tilde{\lambda} - \lambda)(\operatorname{div} \mathbf{u}_3)(\operatorname{div} \mathbf{u}_2)\, dx. \tag{3.1}$$

To show this, let $\mathbf{u}_2$ be a fixed solution to $L_{D_2}\mathbf{u}_2 = 0$ in a vicinity of the set $\overline{D_1 \cup D_2}$ and let us denote by H the class of all functions $\mathbf{u}_3$ which are defined on open sets containing $\overline{D_1 \cup D_2}$ and satisfy the identity (3.1). Then we have only to show that if $\mathbf{u}_3$ is a solution to $L_{D_3}\mathbf{u}_3 = 0$ near the set $\overline{D_1 \cup D_2}$, then $\mathbf{u}_3$ belongs to H.

Extend $\mathbf{u}_2$ to a function $\mathbf{U}_2$ in $H_0^1(\Omega)$.

Suppose that $\mathbf{u} \in H^1(\Omega)$ is a solution to $L_{D_1}\mathbf{u} = 0$ in Ω. We shall show that $\mathbf{u} \in H$. Let $\mathbf{v} \in H^1(\Omega)$ be the solution to the Dirichlet problem

$$L_{D_2}\mathbf{v} = 0 \text{ in } \Omega \qquad \text{and} \qquad \mathbf{v} = \mathbf{u} \text{ on } \partial\Omega,$$

and set $\mathbf{w} = \mathbf{v} - \mathbf{u}$.

Then $\mathbf{w} = 0$ on $\partial\Omega$ and $\mu\Delta\mathbf{w} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{w} = 0$ in $\Omega \setminus \overline{D_1 \cup D_2}$. Morover by the hypothesis of theorem,

$$\mu(\nabla\mathbf{w} + \nabla\mathbf{w}^t)N + \lambda(\operatorname{div}\mathbf{w})N = 0 \qquad \text{on } \partial\Omega.$$

Hence it follows from the unique continuation theorem that

$$\mathbf{w} = 0 \qquad \text{on } \Omega \setminus \overline{D_1 \cup D_2}.$$

Note that since $L_{D_1}\mathbf{u} = 0$ and $L_{D_2}\mathbf{v} = 0$ in Ω, for all $\boldsymbol{\xi} \in H_0^1(\Omega)$

$$\begin{aligned} &\int_\Omega \mu_{D_2}(\nabla\mathbf{w} + \nabla\mathbf{w}^t)\nabla\boldsymbol{\xi} + \lambda_{D_2}(\operatorname{div}\mathbf{w})(\operatorname{div}\boldsymbol{\xi}) \\ &\qquad = \int_\Omega (\mu_{D_1} - \mu_{D_2})(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\boldsymbol{\xi} + \left(\lambda_{D_1} - \lambda_{D_2}\right)(\operatorname{div}\mathbf{u})(\operatorname{div}\boldsymbol{\xi}). \end{aligned} \tag{3.2}$$

Then since $L_{D_2}\mathbf{u}_2 = 0$ near $\overline{D_1 \cup D_2}$ and $\mathbf{w} = 0$ outside $D_1 \cup D_2$, taking $\boldsymbol{\xi} = \mathbf{U}_2$ in (3.2), we have

$$0 = \int_\Omega (\mu_{D_1} - \mu_{D_2})(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{U}_2 + \left(\lambda_{D_1} - \lambda_{D_2}\right)(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{U}_2)$$

or equivalently

$$\begin{aligned} &\int_{D_1} (\tilde{\mu} - \mu)(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{u}_2 + (\tilde{\lambda} - \lambda)(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{u}_2)\, dx \\ &\qquad = \int_{D_2} (\tilde{\mu} - \mu)(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{u}_2 + (\tilde{\lambda} - \lambda)(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{u}_2)\, dx \end{aligned}$$

and thus $\mathbf{u} \in H$.

Suppose next that $\mathbf{u}$ is a solution to $L_{D_1}\mathbf{u} = 0$ near $\overline{D_1 \cup D_2}$. Then by Lemma 2.3, we can choose a sequence $\{\mathbf{u}_m\}$ in $H^1(\Omega)$ such that (i) $L_{D_1}\mathbf{u}_m = 0$ in Ω and (ii) $\mathbf{u}_m \to \mathbf{u}$ in $H^1(D_1 \cup D_2)$. Since each $\mathbf{u}_m$ satisfies (3.1), so does $\mathbf{u}$ and $\mathbf{u} \in H$.

Now we prove our claim. Let $\mathbf{u}_3$ be a solution to $L_{D_3}\mathbf{u}_3 = 0$ in a smooth subdomain Ω' of Ω containing $\overline{D_1 \cup D_2}$. Consider a sequence $\{\Omega_m\}$ of subdomains of Ω defined by $\Omega_m = D_1 \cup D_2 \cup \{X \in B_R : \operatorname{dist}(X, \partial D_1) < R/m\}$.

Define

$$\mu_m = \begin{cases} \mu & \text{in } \Omega_m \setminus (D_1 \cup D_2) \\ \mu_{D_3} & \text{in } \Omega \setminus (\Omega_m \setminus (D_1 \cup D_2)) \end{cases}$$

and

$$\lambda_m = \begin{cases} \lambda & \text{in } \Omega_m \setminus (D_1 \cup D_2) \\ \lambda_{D_3} & \text{in } \Omega \setminus (\Omega_m \setminus (D_1 \cup D_2)). \end{cases}$$

Then it is easy to see that

$$\left| \{ \mu_m \neq \mu_{D_3} \text{ or } \lambda_m \neq \lambda_{D_3} \} \right| \to 0 \qquad \text{as } m \to +\infty.$$

Hence if $\mathbf{u}_{3,m} \in H^1(\Omega)$ is the solution to the Dirichlet problem

$$L_m \mathbf{u}_{3,m} = 0 \text{ in } \Omega' \qquad \text{and} \qquad \mathbf{u}_{3,m} = \mathbf{u}_3 \text{ on } \partial\Omega'.$$

where L_m denotes the operator corresponding to the system of elasticity with Lamé constants μ_m and λ_m, then by Lemma 2.2, $\mathbf{u}_{3,m} \to \mathbf{u}_3$ in $H^1(\Omega')$.

Note that $\mu_m = \mu_{D_1}$ and $\lambda_m = \lambda_{D_1}$ near $\overline{D_1}$. Thus each $\mathbf{u}_{3,m}$ belongs to H and consequently, $\mathbf{u}_3$ also belongs to H. This completes the proof of Claim 1. We now return to the proof of Theorem 0.1. We shall denote $\mathcal{S}_j = \mathcal{S}_{D_j}$ and $\tilde{\mathcal{S}}_j = \tilde{\mathcal{S}}_{D_j}$ for $j = 2, 3$. Let $\{X_m\}$ be a sequence in $B_{R/4} \setminus \overline{D_1}$ with $X_m \to Q$. Then by using Theorem 1.2 we can choose sequences $\{\mathbf{f}_{j,m}\}$ and $\{\mathbf{g}_{j,m}\}$ in $L^2(\partial D_j)$ for $j = 2, 3$ such that

$$\begin{cases} \mathcal{S}_2 \mathbf{f}_{2,m} - \tilde{\mathcal{S}}_2 \mathbf{g}_{2,m} = -\Gamma_1(X_m - .) \\ \dfrac{\partial}{\partial \nu} \mathcal{S}_2 \mathbf{f}_{2,m}^- - \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_2 \mathbf{g}_{2,m}^+ = -\dfrac{\partial}{\partial \nu} \Gamma_1(X_m - .) \end{cases} \qquad \text{on } \partial D_2,$$

and

$$\begin{cases} \mathcal{S}_3 \mathbf{f}_{3,m} - \tilde{\mathcal{S}}_3 \mathbf{g}_{3,m} = \tilde{\Gamma}_1(X_m - .) \\ \dfrac{\partial}{\partial \nu} \mathcal{S}_3 \mathbf{f}_{3,m}^- - \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_3 \mathbf{g}_{3,m}^+ = \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\Gamma}_1(X_m - .) \end{cases} \qquad \text{on } \partial D_3,$$

where $\Gamma_1(X)$ and $\tilde{\Gamma}_1(X)$ are the first columns of the matrices $\Gamma(X)$ and $\tilde{\Gamma}(X)$, respectively. Then we define

$$\mathbf{K}_{2,m}(Y) = \begin{cases} \Gamma_1(X_m - Y) + \mathcal{S}_2 \mathbf{f}_{2,m}(Y), & Y \notin D_2 \\ \tilde{\mathcal{S}}_2 \mathbf{g}_{2,m}(Y), & Y \in D_2 \end{cases}$$

and

$$\mathbf{K}_{3,m}(Y) = \begin{cases} \mathcal{S}_3\mathbf{f}_{3,m}(Y), & Y \notin D_3 \\ \tilde{\Gamma}_1(X_m - Y) + \tilde{\mathcal{S}}_3\mathbf{g}_{3,m}(Y), & Y \in D_3. \end{cases}$$

It is easy to check that each function $\mathbf{K}_{j,m}$ satisfies $L_{D_j}\mathbf{K}_{j,m} = 0$ near the set $\overline{D_1 \cup D_2}$. Hence it follows from (3.1) that

$$\begin{aligned} &\int_{D_1} (\tilde{\mu} - \mu)\big(\nabla \mathbf{K}_{3,m} + \nabla \mathbf{K}_{3,m}^t\big)\nabla \mathbf{K}_{2,m} \\ &\qquad + (\tilde{\lambda} - \lambda)(\operatorname{div} \mathbf{K}_{3,m})(\operatorname{div} \mathbf{K}_{2,m})\, dY \\ &\qquad = \int_{D_2} (\tilde{\mu} - \mu)\big(\nabla \mathbf{K}_{3,m} + \nabla \mathbf{K}_{3,m}^t\big)\nabla \mathbf{K}_{2,m} \\ &\qquad\quad + (\tilde{\lambda} - \lambda)(\operatorname{div} \mathbf{K}_{3,m})(\operatorname{div} \mathbf{K}_{2,m})\, dY \end{aligned}$$

and therefore

$$\begin{aligned} &\int_{B_R \cap D_1} (\tilde{\mu} - \mu)\big(\nabla \tilde{\Gamma}_1(X_m - Y) + \nabla \tilde{\Gamma}_1(X_m - Y)^t\big)\nabla \Gamma_1(X_m - Y) \\ &\quad + (\tilde{\lambda} - \lambda)\big(\operatorname{div} \tilde{\Gamma}_1(X_m - Y)\big)(\operatorname{div} \Gamma_1(X_m - Y))\, dY \\ &\quad = I_m^1 - I_m^2 - I_m^3 - I_m^4 - I_m^5, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} I_m^1 = &\int_{D_2 \setminus D_1} (\tilde{\mu} - \mu)\big(\nabla \mathcal{S}_3\mathbf{f}_{3,m} + \nabla \mathcal{S}_3\mathbf{f}_{3,m}^t\big)\nabla \tilde{\mathcal{S}}_2\mathbf{g}_{2,m} \\ &+ (\tilde{\lambda} - \lambda)(\operatorname{div} \mathcal{S}_3\mathbf{f}_{3,m})\big(\operatorname{div} \tilde{\mathcal{S}}_2\mathbf{g}_{2,m}\big), \\ I_m^2 = &\int_{(D_1 \setminus D_2)\setminus B_R} (\tilde{\mu} - \mu)\big(\nabla \tilde{\Gamma}_1(X_m - Y) + \nabla \tilde{\Gamma}_1(X_m - Y)^t\big)\nabla \Gamma_1(X_m - Y) \\ &+ (\tilde{\lambda} - \lambda)\big(\operatorname{div} \tilde{\Gamma}_1(X_m - Y)\big)(\operatorname{div} \Gamma_1(X_m - Y))\, dY, \\ I_m^3 = &\int_{D_1 \setminus D_2} (\tilde{\mu} - \mu)\big(\nabla \tilde{\Gamma}_1(X_m - Y) + \nabla \tilde{\Gamma}_1(X_m - Y)^t\big)\nabla \mathcal{S}_2\mathbf{f}_{2,m}(Y) \\ &+ (\tilde{\lambda} - \lambda)\big(\operatorname{div} \tilde{\Gamma}_1(X_m - Y)\big)(\operatorname{div} \mathcal{S}_2\mathbf{f}_{2,m}(Y))\, dY, \\ I_m^4 = &\int_{D_1 \setminus D_2} (\tilde{\mu} - \mu)\big(\nabla \tilde{\mathcal{S}}_3\mathbf{g}_{3,m}(Y) + \nabla \tilde{\mathcal{S}}_3\mathbf{g}_{3,m}(Y)^t\big)\nabla \Gamma_1(X_m - Y) \\ &+ (\tilde{\lambda} - \lambda)\big(\operatorname{div} \tilde{\mathcal{S}}_3\mathbf{g}_{3,m}(Y)\big)(\operatorname{div} \Gamma_1(X_m - Y))\, dY \end{aligned}$$

and

$$I_m^5 = \int_{D_1 \setminus D_2} (\tilde{\mu} - \mu)\left(\nabla \tilde{\mathcal{S}}_3 \mathbf{g}_{3,m} + \nabla \tilde{\mathcal{S}}_3 \mathbf{g}_{3,m}{}^t\right) \nabla \mathcal{S}_2 \mathbf{f}_{2,m}$$

$$+ (\tilde{\lambda} - \lambda)\left(\operatorname{div} \tilde{\mathcal{S}}_3 \mathbf{g}_{3,m}\right)(\operatorname{div} \mathcal{S}_2 \mathbf{f}_{2,m}).$$

Note that

$$D_k \Gamma^{ij}(X) = \frac{1}{\omega_n}\left[-A\delta_{ij} X_k + B\left(\delta_{ik} X_j + \delta_{jk} X_i - n X_i X_j X_k |X|^{-2}\right)\right]|X|^{-n}$$

and

$$D_k \tilde{\Gamma}^{ij}(X) = \frac{1}{\omega_n}\left[-\tilde{A}\delta_{ij} X_k + \tilde{B}\left(\delta_{ik} X_j + \delta_{jk} X_i - n X_i X_j X_k |X|^{-2}\right)\right]|X|^{-n}.$$

Then we see that if $X \in B_{R/4}$ and $P \in R^n \setminus B_{R/2}$, then

$$\left|D_k \Gamma^{ij}(X - P)\right| \leq \frac{C}{R^{n-1}} \qquad \text{and} \qquad \left|D_k \tilde{\Gamma}^{ij}(X - P)\right| \leq \frac{C}{R^{n-1}}.$$

Combining these estimates with Theorem 1.2, we see that

$$\|\mathbf{f}_{2,m}\|_{L^2(\partial D_2)} + \|\mathbf{g}_{2,m}\|_{L^2(\partial D_2)} + \|\mathbf{f}_{3,m}\|_{L^2(\partial D_3)} + \|\mathbf{g}_{3,m}\|_{L^2(\partial D_3)}$$

$$\leq C\Bigg[\|\Gamma_1(X_m - \cdot)\|_{L^2(\partial D_2)} + \left\|\frac{\partial}{\partial \nu}\Gamma_1(X_m - \cdot)\right\|_{L^2(\partial D_2)}$$

$$+ \|\tilde{\Gamma}_1(X_m - \cdot)\|_{L^2(\partial D_3)} + \left\|\frac{\partial}{\partial \tilde{\nu}}\tilde{\Gamma}_1(X_m - \cdot)\right\|_{L^2(\partial D_3)}\Bigg] \leq C,$$

where the last constant C is independent of m. Hence by means of the estimate (1.3), it can be easily proved that I_m' s are uniformly bounded on m.

Therefore to complete the proof of Theorem 0.1, it suffices to show that the identity (3.3) is not possible by showing the following claim:

Claim 2. If we set

$$I_m = \int_{B_R \cap D_1} (\tilde{\mu} - \mu)\left(\nabla \Gamma_1(X_m - Y) + \nabla \Gamma_1(X_m - Y)^t\right) \nabla \tilde{\Gamma}_1(X_m - Y)$$

$$+ (\tilde{\lambda} - \lambda)(\operatorname{div} \Gamma_1(X_m - Y))\left(\operatorname{div} \tilde{\Gamma}_1(X_m - Y)\right) dY,$$

then

$$|I_m| \to \infty \qquad \text{as} \qquad m \to \infty.$$

We prove this claim by direct calculations. Remembering the formulas for $D_k \tilde{\Gamma}^{ij}$ and $D_k \tilde{\Gamma}^{ij}$, we calculate to obtain

$$\nabla \tilde{\Gamma}_1(X) \nabla \Gamma_1(X) |X|^{2n}$$
$$= (A\tilde{A} + B\tilde{B})|X|^2 + \left\{(A\tilde{B} + \tilde{A}B)(n-2) + B\tilde{B}(n-1)(n-2)\right\} X_1^2,$$
$$\nabla \tilde{\Gamma}_1(X)^t \nabla \Gamma_1(X) |X|^{2n}$$
$$= -(A\tilde{B} + \tilde{A}B)|X|^2 + \left\{A\tilde{A} + (A\tilde{B} + \tilde{A}B)(n-1) + B\tilde{B}(n-2)^2\right\} X_1^2,$$
$$\operatorname{div} \Gamma_1(X) = \frac{-X_1|X|^{-n}}{\omega_n(2\mu + \lambda)} \qquad \text{and} \qquad \operatorname{div} \tilde{\Gamma}_1(X) = \frac{-X_1|X|^{-n}}{\omega_n\left(2\tilde{\mu} + \tilde{\lambda}\right)}$$

so that

$$\left(\nabla \tilde{\Gamma}_1(X) + \nabla \tilde{\Gamma}_1(X)^t\right) \nabla \Gamma_1(X) = (A - B)\left(\tilde{A} - \tilde{B}\right)|X|^{2-2n}$$
$$+ \left\{A\tilde{A} + (A\tilde{B} + \tilde{A}B)(2n-3) + B\tilde{B}(n-2)(2n-3)\right\} X_1^2 |X|^{-2n}$$

and

$$\left(\operatorname{div} \tilde{\Gamma}_1(X)\right)\left(\operatorname{div} \Gamma_1(X)\right) = \frac{X_1^2|X|^{-2n}}{\omega_n(2\mu + \lambda)\left(2\tilde{\mu} + \tilde{\lambda}\right)}.$$

Hence the integrand of $|I_m|$ is of the form $r(X_m - Y)|X_m - Y|^{2-2n}$ where $r(X)$ is a function with $r(X) \geq \epsilon > 0$ and thus

$$|I_m| = \int_{B_R \cap D_1} r(X_m - Y)|X_m - Y|^{2-2n} \, dY \to \infty \qquad \text{as } m \to \infty.$$

This completes the proof of Claim 2 and so the proof of Theorem 0.1.

Remark. If $\Omega \setminus \overline{D_1 \cup D_2}$ is not connected, the identity (3.1) holds for solution u_j $(j = 2, 2)$ of $L_{D_j} u_j = 0$ in vicinities of $\Omega \setminus \Omega_1$ where Ω_1 is the connected component of $\Omega \setminus \overline{D_1 \cup D_2}$ whose boundary contains $\partial\Omega$.

4. PROOF OF THEOREM 0.2

We prove the theorem by showing that if $\Lambda_{D_1} \boldsymbol{\varphi} = \Lambda_{D_2} \boldsymbol{\varphi}$ for some $\boldsymbol{\varphi} \in H^{1/2}(\partial\Omega)$ but $D_1 \neq D_2$ then $\boldsymbol{\varphi} \in \Phi$, and furthermore if in addition $\mu \neq \tilde{\mu}$ then $\boldsymbol{\varphi} \in \Psi_0$.

Suppose that $\Lambda_{D_1}\boldsymbol{\varphi} = \Lambda_{D_2}\boldsymbol{\varphi}$ and $D_1 \neq D_2$. Without loss of generality, we may assume that $D_1 \subsetneq D_2$. For each $j = 1$ and 2, let $\mathbf{u}_j$ be the solution to the Dirichlet problem: $L_{D_j}\mathbf{u}_j = 0$ in Ω and $\mathbf{u}_j = \boldsymbol{\varphi}$ on $\partial\Omega$, and set $\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$. Then since $\Lambda_{D_1}\boldsymbol{\varphi} = \Lambda_{D_2}\boldsymbol{\varphi}$, we have for all $\mathbf{v} \in H^1(\Omega)$,

$$\int_\Omega \mu_{D_1}(\nabla\mathbf{u}_1 + \nabla\mathbf{u}_1^t)\nabla\mathbf{v} + \lambda_{D_1}(\operatorname{div}\mathbf{u}_1)(\operatorname{div}\mathbf{v})$$
$$= \int_\Omega \mu_{D_2}(\nabla\mathbf{u}_2 + \nabla\mathbf{u}_2^t)\nabla\mathbf{v} + \lambda_{D_2}(\operatorname{div}\mathbf{u}_2)(\operatorname{div}\mathbf{v})$$

and so

$$\int_\Omega \mu_{D_2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{v} + \lambda_{D_2}(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{v})$$
$$= \int_\Omega (\mu_{D_1} - \mu_{D_2})(\nabla\mathbf{u}_1 + \nabla\mathbf{u}_1^t)\nabla\mathbf{v} + (\lambda_{D_1} - \lambda_{D_2})(\operatorname{div}\mathbf{u}_1)(\operatorname{div}\mathbf{v}).$$

Taking $\mathbf{v} = \mathbf{u}$ and $\mathbf{v} = \mathbf{u}_2$, respectively, we obtain

$$\int_\Omega \mu_{D_2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{u} + \lambda_{D_2}(\operatorname{div}\mathbf{u})^2$$
$$= \int_\Omega (\mu_{D_1} - \mu_{D_2})(\nabla\mathbf{u}_1 + \nabla\mathbf{u}_1^t)\nabla\mathbf{u} + (\lambda_{D_1} - \lambda_{D_2})(\operatorname{div}\mathbf{u}_1)(\operatorname{div}\mathbf{u}) \tag{4.1}$$

and

$$\int_\Omega (\mu_{D_1} - \mu_{D_2})(\nabla\mathbf{u}_1 + \nabla\mathbf{u}_1^t)\nabla\mathbf{u}_2 + (\lambda_{D_1} - \lambda_{D_2})(\operatorname{div}\mathbf{u}_1)(\operatorname{div}\mathbf{u}_2)$$
$$= \int_\Omega \mu_{D_2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{u}_2 + \lambda_{D_2}(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{u}_2) = 0. \tag{4.2}$$

The last equality follows from the fact that $L_{D_2}\mathbf{u}_2 = 0$ in Ω and $\mathbf{u} \in H_0^1(\Omega)$. From (4.1) and (4.2),

$$\int_\Omega \mu_{D_2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\nabla\mathbf{u} + \lambda_{D_2}(\operatorname{div}\mathbf{u})^2$$
$$= -\int_\Omega (\mu_{D_1} - \mu_{D_2})(\nabla\mathbf{u}_1 + \nabla\mathbf{u}_1^t)\nabla\mathbf{u}_1 + (\lambda_{D_1} - \lambda_{D_2})(\operatorname{div}\mathbf{u}_1)^2$$

or

$$0 = \int_\Omega \frac{\mu_{D_2}}{2}|\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 + \lambda_{D_2}(\operatorname{div} \mathbf{u})^2$$

$$+ \int_\Omega \frac{1}{2}(\mu_{D_1} - \mu_{D_2})|\nabla \mathbf{u}_1 + \nabla \mathbf{u}_1^t|^2 + (\lambda_{D_1} - \lambda_{D_2})(\operatorname{div} \mathbf{u}_1)^2. \quad (4.3)$$

By a similar argument, we obtain

$$0 = \int_\Omega \frac{\mu_{D_1}}{2}|\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 + \lambda_{D_1}(\operatorname{div} \mathbf{u})^2$$

$$+ \int_\Omega \frac{1}{2}(\mu_{D_2} - \mu_{D_1})|\nabla \mathbf{u}_2 + \nabla \mathbf{u}_2^t|^2 + (\lambda_{D_2} - \lambda_{D_1})(\operatorname{div} \mathbf{u}_2)^2. \quad (4.4)$$

Now using the monotonicity assumption together with (0.5), we can easily prove that all terms in either (4.3) or (4.4) are nonnegative. Hence using the estimate (1.2), we obtain

$$\int_\Omega |\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 = 0, \quad (4.5)$$

and it follows from Lemma 1.1 and the Poincaré inequality that

$$\mathbf{u} = 0 \qquad \text{in } \Omega$$

and so by (4.3) and (4.4) we obtain

$$\int_{D_2 \setminus \overline{D}_1} \frac{1}{2}(\tilde{\mu} - \mu)|\nabla \mathbf{u}_1 + \nabla \mathbf{u}_1^t|^2 + (\tilde{\lambda} - \lambda)(\operatorname{div} \mathbf{u}_1)^2 = 0 \quad (4.6)$$

and

$$\int_{D_2 \setminus \overline{D}_1} \frac{1}{2}(\tilde{\mu} - \mu)|\nabla \mathbf{u}_2 + \nabla \mathbf{u}_2^t|^2 + (\tilde{\lambda} - \lambda)(\operatorname{div} \mathbf{u}_2)^2 = 0. \quad (4.7)$$

Claim 1. If $\mu = \tilde{\mu}$, then $\boldsymbol{\varphi} \in \Phi$.

Suppose that $\mu = \tilde{\mu}$. Then since $\lambda \neq \tilde{\lambda}$ by (0.3), from the identities (4.6) and (4.7) we obtain

$$\operatorname{div} \mathbf{u}_1 = \operatorname{div} \mathbf{u}_2 = 0 \qquad \text{in } D_2 \setminus \overline{D}_1.$$

Note that $\mathbf{u}_1$ is analytic in $\Omega \setminus \overline{D}_1$ and $\mathbf{u}_2$ is analytic in D_2. Hence it follows from the analytic continuation that

$$\operatorname{div} \mathbf{u}_1 = 0 \qquad \text{in } \Omega \setminus \overline{D}_1 \qquad \text{and} \qquad \operatorname{div} \mathbf{u}_2 = 0 \qquad \text{in } D_2.$$

Since $\mathbf{u}_1 = \mathbf{u}_2$, we conclude that

$$\operatorname{div} \mathbf{u}_1 = \operatorname{div} \mathbf{u}_2 = 0 \qquad \text{in } \Omega,$$

which implies then that

$$\Delta \mathbf{u}_j = 0 \qquad \text{in } \Omega \qquad \text{for each } j = 1, 2.$$

This proves that $\boldsymbol{\varphi} = \mathbf{u}_1|_{\partial\Omega} \in \Phi$.

Claim 2. If $\mu \neq \tilde{\mu}$, then $\boldsymbol{\varphi} \in \Psi_0$.

Suppose that $\mu \neq \tilde{\mu}$. Then by the assumption (0.5) and the estimate (1.2), it follows from (4.6) and (4.7) that

$$\nabla \mathbf{u}_1 + \nabla \mathbf{u}_1^t = \nabla \mathbf{u}_2 + \nabla \mathbf{u}_2^t = 0 \qquad \text{in } D_2 \setminus \overline{D}_1.$$

Therefore using the same argument as above, we obtain

$$\nabla \mathbf{u}_1 + \nabla \mathbf{u}_1^t = \nabla \mathbf{u}_2 + \nabla \mathbf{u}_2^t = 0 \qquad \text{in } \Omega$$

which implies that $\boldsymbol{\varphi} = \mathbf{u}_1|_{\partial\Omega} \in \Psi_0$.

This completes the whole proof of Theorem 0.2.

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